# DESIGN OF A TRANSVERSELY LAYERED CANTILEVER OF MINIMUM MASS
# UNDER SPECIFIED MAXIMUM-DEFLECTION CONSTRAINT

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*The problem of optimal design of a transversely layered cantilever beam of minimum mass composed of a finite set of elastic homogeneous isotropic materials is considered for the case where a constraint is imposed on the maximum deflection of the beam and information on the loads applied to the beam is incomplete. It is shown that, among these loads, there exists the "worst" point load whose magnitude is equal to the resultant of the forces applied to the beam, for which the deflection is maximal for any material distribution along the beam. The necessary optimality conditions are obtained for the mass-minimization problem of the beam under the "worst" loading, a numerical algorithm is developed for the synthesis of the optimal cantilever, and a numerical example is given.*

**Key words:** *layered material, beam, optimal design and control, needle variations, maximum principle.*

**Introduction.** Formulations and algorithms for solving optimization problems of various layered structural elements under wave, thermal, and mechanical loads were proposed in [1–4]. In the present paper, the optimal design problem for a transversely layered beam of minimum mass and constant cross section, composed from a finite set of materials, is considered using the method developed in [3]. Previously, the shape-optimization problems for a homogeneous beam as the simplest structural element were considered [5]. The resulting designs are characterized by abrupt variations in the height or the width of the cross section along the beam, whereas it is well known that the one-dimensional mathematical model of the bar is adequate only for small gradients of variation in the shape function.

**1. Formulation of the Problem.** Let there be a set $W$ consisting of $k$ elastic homogeneous isotropic materials. Given this set, it is required to synthesize a transversely layered cantilever of minimum mass under a constraint imposed on the maximum deflection of the beam.

We consider a beam of length $l$ and constant cross-sectional area $S$ whose left end is rigidly clamped and whose right end is free. We use the Cartesian coordinate system $(x, y, z)$ with its origin located at the left end of the beam and the $x$ axis directed along the beam. The beam is subjected to external transverse loads $f(x)$. Moreover, at the point $x = x_c$ $(0 < x_c < l)$, the load $q(x)$, which is a point force $\boldsymbol{P}$, is applied, i.e., $q(x) = P\delta(x - x_c)$ (Fig. 1). Here $\delta$ is the Dirac delta function. The cantilever is bent in the $(x, z)$ plane. It is assumed that the distributed load $f(x)$ and the point load $q(x)$ are nonnegative everywhere and have a resultant that does not exceed a specified value of $P_0$, i.e.,

$$f(x) \geqslant 0, \qquad q(x) \geqslant 0, \qquad \int\limits_0^l f(x)\, dx + P \leqslant P_0. \tag{1.1}$$

Let $\sigma_0$ and $\rho_0$ be the characteristic scales of the stress and density, respectively. We introduce the following dimensionless variables (below, the asterisk indicating dimensionless quantities is omitted):
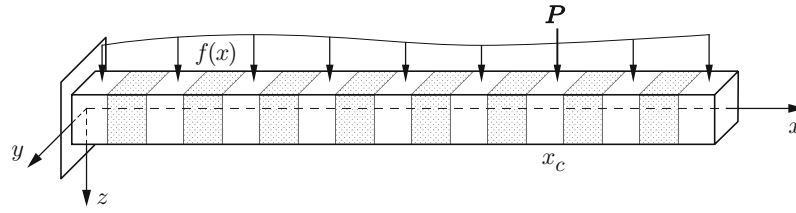
Fig. 1. Transversely layered cantilever under a continuously distributed load $f(x)$ and point load $\boldsymbol{P}$.

$$x^* = \frac{x}{l}, \quad w^* = \frac{w}{l}, \quad E^* = \frac{E}{\sigma_0}, \quad \rho^* = \frac{\rho}{\rho_0}, \quad f^* = \frac{fl^3}{\sigma_0 J},$$

$$P^* = \frac{Pl^2}{\sigma_0 J}, \quad P_0^* = \frac{P_0 l^2}{\sigma_0 J}, \quad Q^* = \frac{Ql^2}{\sigma_0 J}, \quad M^* = \frac{Ml}{\sigma_0 J}. \tag{1.2}$$

Here $w(x)$ is the beam deflection, $E(x)$ and $\rho(x)$ are the Young moduli and densities of the materials of the layers, respectively, $J$ is the moment of inertia of the cross-sectional area $S$ of the beam about the $y$ axis, and $M(x)$ and $Q(x)$ are the bending moment and the transverse shear force, respectively.

In the variables (1.2), the cantilever equilibrium equations and the boundary conditions are

$$M'(x) = Q(x), \qquad Q'(x) = -f(x) - q(x), \qquad M(x) = -E(x)w''(x); \tag{1.3}$$

$$w(0) = w'(0) = M(1) = Q(1) = 0, \tag{1.4}$$

where the prime denotes differentiation with respect to the $x$ coordinate. If $x_c = 1$, then $q(x) = 0$ in Eqs. (1.3) and $Q(1) = P$ in the boundary conditions (1.4).

At the interfaces between the layers $x_i \in (0,1)$ where the Young moduli of the materials are discontinuous, the following continuity conditions are specified: continuous deflection $w$, rotation $w'$, bending moment $M$, and transverse shear force $Q$ [in the absence of a point load from set (1.1) at the point $x_i$]:

$$[w(x_i)]\Big|_-^+ = [w'(x_i)]\Big|_-^+ = [M(x_i)]\Big|_-^+ = [Q(x_i)]\Big|_-^+ = 0. \tag{1.5}$$

We introduce the piecewise constant function

$$\alpha(x) = \{\alpha_j\colon x_{j-1} < x \leqslant x_j, \ j = 1, \ldots, n, \ x_0 = 0, \ x_n = 1\}, \tag{1.6}$$

which characterizes the structure of the transversely layered cantilever (the number, dimensions, and materials of the layers). The values of $\alpha_j$ belong to the finite discrete set

$$\alpha_j \in U = \{1, \ldots, k\}, \tag{1.7}$$

which corresponds to the original set of materials $W$, and all characteristics of the materials from this set are functions of the distribution $\alpha(x)$ on the interval $[0,1]$. In the optimization problem considered, the function $\alpha(x)$ is taken as the control.

The problem of optimal design of the cantilever is formulated as follows. Among the piecewise constant functions $\alpha(x)$ (1.6) with the range of values $U$ (1.7) for any realization of the external loads $f(x)$ and $q(x)$ from set (1.1), it is required to find the control $\alpha_{\mathrm{opt}}(x)$ that ensures a minimum of the mass functional

$$F_0(\alpha) = \int\limits_0^1 \rho(x, \alpha)\, dx \tag{1.8}$$

under the constraint imposed on the cantilever deflection

$$F_1(\alpha) = \max_{x \in [0,1]} |w(x, \alpha)| - w_0 \leqslant 0, \tag{1.9}$$

where $w_0$ is a specified quantity.

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Since the applied load is indeterminate, the optimization problem formulated above belongs to the so-called problems with incomplete information [5].

Before solving problem (1.1)–(1.9), we study the properties of the function $w(x)$. Let there be a certain distribution $\alpha(x)$ of materials from the set $W$ along the cantilever. We compare the expression for the deflection $w(x)$ for arbitrary distributed external loads $f(x)$ and $q(x)$ from set (1.1) with that for the point load $P_0$ applied to the end of the cantilever.

We integrate Eqs. (1.3), taking into account the boundary conditions (1.4). As a result, we obtain

$$Q(x) = \begin{cases} \int\limits_x^1 f(t)\, dt, & x > x_c \in (0,1), \\[2em] \int\limits_x^1 f(t)\, dt + P, & x \leqslant x_c \in (0,1]. \end{cases} \tag{1.10}$$

For the point load $P_0$ applied to the end of the cantilever, the beam equilibrium equations and the boundary conditions are

$$\tilde{M}'(x) = \tilde{Q}(x), \qquad \tilde{Q}'(x) = 0, \qquad \tilde{M}(x) = -E(x)\tilde{w}''(x); \tag{1.11}$$

$$\tilde{w}(0) = \tilde{w}'(0) = \tilde{M}(1) = 0, \qquad \tilde{Q}(1) = P_0. \tag{1.12}$$

Thus, $\tilde{Q}(x) = P_0$ and $x \in [0,1]$. By virtue of conditions (1.1) and expressions (1.10) and (1.11), we obtain $\tilde{Q}(x) \geqslant Q(x) \geqslant 0$ and $\tilde{M}'(x) \geqslant M'(x) \geqslant 0$. Since $M(1) = \tilde{M}(1) = 0$, then $\tilde{M}(x) \leqslant M(x) \leqslant 0$, $x \in [0,1]$, i.e., $\tilde{w}''(x) \geqslant w''(x) \geqslant 0$. Integrating the last inequality with allowance for the boundary conditions (1.4) and (1.12), we find that $\tilde{w}'(x) \geqslant w'(x) \geqslant 0$. Consequently, the functions $w(x)$ and $\tilde{w}(x)$ increase monotonically within the interval $[0,1]$ and reach a maximum value for $x = 1$. In this case, $\tilde{w}(1) \geqslant w(1)$.

Thus, the point force $P_0$ applied to the end of the cantilever is the "worst" load from set (1.1), for which the deflection is maximal for any material distribution along the beam. Minimization of the cantilever mass is performed for the "worst" load. In this case, the maximum deflection constraint (1.9) reduces to the condition

$$F_1(\alpha) = w(1, \alpha) - w_0 \leqslant 0. \tag{1.13}$$

**2. Necessary Optimality Conditions.** To obtain the necessary optimality conditions in the optimization problem considered, it is required to express the variations of the objective functional (1.8) and constraint (1.13) in terms of the variation of the control $\alpha(x)$.

The matching conditions (1.5) on the interfaces between the layers of the cantilever allow one to introduce the following phase variables continuous within the interval $[0,1]$:

$$\boldsymbol{y}(x) = (y_1, y_2, y_3, y_4)^{\mathrm{t}} = (w, w', M, Q)^{\mathrm{t}}.$$

In the new variables, the controlled system (1.11), (1.12) with the "worst" load from set (1.1) (point force $P_0$ at the point $x = 1$) becomes

$$\boldsymbol{y}'(x) = A(x, \alpha)\boldsymbol{y}(x); \tag{2.1}$$

$$y_1(0) = y_2(0) = y_3(1) = 0, \qquad y_4(1) = P_0. \tag{2.2}$$

Here, the nonzero components $a_{ij}$ of the matrix $A(x, \alpha)$ are given by

$$a_{12} = a_{34} = 1, \qquad a_{23} = -1/E(x).$$

Let $\alpha(x)$ be an admissible control from set (1.7), satisfying constraint (1.13). We consider the perturbed control $\alpha^*(x)$ [6]:

$$\alpha^*(x) = \begin{cases} \theta(x), & x \in M, \quad \theta(x) \in U, \\ \alpha(x), & x \notin M, \quad \mathrm{mes}\, M \ll 1 \end{cases} \tag{2.3}$$

($M \subset [0,1]$ is the small-measure set). The variation of the objective functional $F_0(\alpha)$ (1.8) can be written as

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$$\delta F_0(M, \theta) = \int\limits_M \{\rho(x, \theta) - \rho(x, \alpha)\}\, dx. \tag{2.4}$$

To obtain variation of constraint (1.13), we express the maximum deflection $w(1, \alpha)$ in terms of the phase variables $y_i(x)$ and control $\alpha(x)$. From system (2.1) and boundary conditions (2.2), we obtain

$$w(1, \alpha) = y_1(1) = \frac{1}{P_0} \int\limits_0^1 \frac{y_3^2(x)}{E(\alpha)}\, dx. \tag{2.5}$$

Using expressions (2.1) and (2.2) written in variations

$$\delta \boldsymbol{y}'(x) = A(x, \alpha)\delta \boldsymbol{y}(x) + [A(x, \alpha^*) - A(x, \alpha)]\boldsymbol{y}(x),$$

$$\delta y_1(0) = 0, \quad \delta y_2(0) = 0, \quad \delta y_3(1) = 0, \quad \delta y_4(1) = 0,$$

and taking into account Eq. (2.5), we write the principal part of the increment of the functional $F_1(\alpha)$ (1.13) as

$$\delta F_1(M, \theta) = \delta y_1(1) = \frac{1}{P_0} \int\limits_M y_3^2(x)\Big(\frac{1}{E(\theta)} - \frac{1}{E(\alpha)}\Big)dx. \tag{2.6}$$

We construct the extended functional

$$I(\alpha) = F_0(\alpha) + \lambda(F_1(\alpha) + \xi^2), \tag{2.7}$$

where $\lambda$ and $\xi$ are the Lagrange multiplier and the penalty variable, respectively, which allow for constraint (1.13) [7]. Using expressions (2.4) and (2.6), we write the variation of the functional $I(\alpha)$ (2.7) as

$$\delta I(M, \theta) = \int\limits_M \Big[H(x, \alpha, \boldsymbol{y}) - H(x, \theta, \boldsymbol{y})\Big]\, dx + 2\lambda\xi\, \delta\xi; \tag{2.8}$$

$$H(x, \alpha, \boldsymbol{y}) = -\rho(x, \alpha) - \lambda\, \frac{y_3^2(x)}{P_0 E(\alpha)}. \tag{2.9}$$

The expression (2.9) for the function $H(x, \alpha, \boldsymbol{y})$ implies that the Hamilton function of the optimization problem considered does not contain conjugate variables.

If the control $\alpha(x)$ is optimal (minimizing), the necessary optimality condition $\delta I(M, \theta) \geqslant 0$ should hold for any admissible control $\alpha^*(x)$ (2.3). Then, since the variation $\delta\xi$ is arbitrary and the small-measure set $M$ can be located densely everywhere on the interval $[0, 1]$, relation (2.8) implies that the following conditions are satisfied for the optimal control $\alpha_{\mathrm{opt}}(x)$:

$$\lambda\Big[w(1, \alpha_{\mathrm{opt}}) - w_0\Big] = 0, \qquad \lambda \geqslant 0; \tag{2.10}$$

$$H(x, \alpha_{\mathrm{opt}}, \boldsymbol{y}) = \max_{\theta \in U} H(x, \theta, \boldsymbol{y}). \tag{2.11}$$

Expression (2.10) is the condition of supplementary nonstiffness and sign agreement [7].

Based on relations (2.10) and (2.11) implied by the necessary optimality conditions $\delta I(M, \theta) \geqslant 0$, we formulate the maximum principle for the problem considered.

Let $\alpha_{\mathrm{opt}}(x)$ be the optimal control in problem (1.6)–(1.8), (1.13), (2.1), (2.2) for which system (2.1) subject to the boundary conditions (2.2) has a nontrivial solution $\boldsymbol{y}(x)$. The Hamilton function $H(x, \theta, \boldsymbol{y})$ (2.9) based on this solution reaches a maximum with respect to the argument $\theta$ for the optimal control $\alpha_{\mathrm{opt}}(x)$ for almost all $x \in [0, 1]$, i.e.,

$$H(x, \alpha_{\mathrm{opt}}, \boldsymbol{y}) = \max_{\theta \in U} H(x, \theta, \boldsymbol{y}).$$

In this case, relation (2.10) should hold on the optimal trajectory $\boldsymbol{y}(x)$.

Thus, the optimal control $\alpha_{\mathrm{opt}}(x)$ and the corresponding optimal trajectory $\boldsymbol{y}(x)$ should satisfy the boundary-value problem (2.1), (2.2), constraints (1.13) and (2.10), and the maximum principle (2.11) for almost all $x \in [0, 1]$.

TABLE 1

| Material | $\rho$ | $E$ |
|---|---|---|
| Spheroplastic | 0.65 | 270 |
| Duralumin | 2.85 | 7100 |
| Titanium alloy | 4.60 | 12,000 |
| Steel | 7.80 | 21,000 |
| Copper | 8.93 | 11,200 |

**3. Computational Algorithm.** Since, for the problem considered, the Hamilton function $H(x, \alpha, \boldsymbol{y})$ (2.9) does not contain conjugate variables, the computational algorithm is as follows. Using a uniform grid of nodes $\{x_i\}$, we divide the interval $[0, 1]$ into a reasonably large number of segments of small length $h = x_i - x_{i-1}$ modeling small-measure sets $M_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$.

Let a certain control $\alpha^s(x)$ from the admissible range (1.7), (1.13) be known. For the segment $M_i$, the phase-variable vector $\boldsymbol{y}(x)$ is characterized by its value at the point $x = x_{i-1} + h/2$. The algorithm for finding the $(s + 1)$th approximation includes the following steps:

1. Solve system (2.1), (2.2) and determine the phase-variable vector $\boldsymbol{y}(x)$ at the points $x = x_{i-1} + h/2$ $(i = 1, \ldots, n)$ and the maximum deflection $w(1, \alpha^s)$.

2. In accordance with expression (2.3), specify a new control $\theta_i^*$ on the segment $M_i$ that ensures a minimum variation $\delta F_0(M_i, \theta_i^*)$ (2.4) for the linearized constraint (1.13)

$$F_1(\alpha^*) \approx F_1(\alpha^s) + \delta F_1(M_i, \theta_i^*) \leqslant 0,$$

which can be combined with (2.6) to give

$$\int_{M_i} \frac{y_3^2(x)}{E(\theta_i^*)} \, dx \leqslant \left[w_0 - w(1, \alpha^s)\right] P_0 + \int_{M_i} \frac{y_3^2(x)}{E(\alpha^s)} \, dx.$$

If $\theta_i^* = \alpha^s(x_{i-1} + h/2)$, perform step 2 for the next segment $M_{i+1}$. If $\theta_i^* \neq \alpha^s(x_{i-1} + h/2)$, go to step 3.

3. Set the next approximation for the control

$$\alpha^{s+1}(x) = \begin{cases} \theta_i^*, & x \in M_i, \quad \theta_i^* \in U, \\ \alpha^s(x), & x \notin M_i. \end{cases}$$

4. Given $\alpha^{s+1}(x)$, go to step 1 and consider the segment $M_{i+1}$.

Thus, the control is improved over the entire interval $[0, 1]$. The process is terminated for the grid considered if the control $\alpha(x)$ remains unchanged for any $i = 1, \ldots, n$. If the solution of the problem is such that the control $\alpha(x)$ takes the same value for two or several neighboring segments $M_i$, these segments are united into a macrolayer. The resulting solution is a local minimum in the problem considered.

**Remark 1.** The set $M_i$ may be either an elementary segment $M_i$ proper of a combination of several segments located at different parts of the interval $[0, 1]$. Varying the control $\alpha(x)$ simultaneously for several elementary segments, one can avoid the deadlock situation [6] where the structure is not yet optimal but cannot, nonetheless, be improved by locally varying the control only for one elementary segment.

**4. Numerical Example.** We consider a set $W$ consisting of five materials whose dimensionless characteristics are given in Table 1. It is required to design a cantilever beam of minimum mass for a given maximum-deflection constraint. The beam is loaded by transverse forces $f(x)$ and $q(x)$ whose resultant is $P_0 = 500$, and the maximum deflection of the beam should not exceed the value $w_0 = 0.02$. The boundary conditions (2.2) are specified on the cantilever ends.

Various material distributions along the beam optimized were used as the initial approximations. As a result, three optimal cantilevers of equal mass $F_0 = 2.962$ but having different maximum deflections were obtained:

— Three-layered beam with the maximum deflection $w(1) = 0.0197$, consisting of a titanium-alloy layer $[0, x_1]$, a Duralumin layer $[x_1, x_2]$, and a spheroplastic layer $[x_2, 1]$ ($x_1 = 0.24$ and $x_2 = 0.86$);

— Five-layered beam with the maximum deflection $w(1) = 0.0198$, consisting of titanium-alloy layers $[0, x_1]$ and $[x_2, x_3]$, Duralumin layers $[x_1, x_2]$ and $[x_3, x_4]$, and a spheroplastic layer $[x_4, 1]$ ($x_1 = 0.14$, $x_2 = 0.16$, $x_3 = 0.26$, and $x_4 = 0.86$);
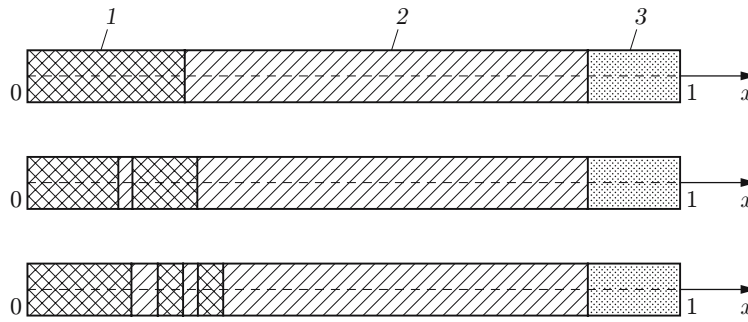
Fig. 2. Optimal cantilevers: titanium-alloy layers (1), Duralumin layers (2), and spheroplastic layers (3).

— Seven-layered beam with the maximum deflection $w(1) = 0.0199$, consisting of titanium-alloy layers $[0, x_1]$, $[x_2, x_3]$, and $[x_4, x_5]$, Duralumin layers $[x_1, x_2]$, $[x_3, x_4]$, and $[x_5, x_6]$, and a spheroplastic layer $[x_6, 1]$ ($x_1 = 0.16$, $x_2 = 0.2$, $x_3 = 0.24$, $x_4 = 0.26$, $x_5 = 0.3$, and $x_6 = 0.86$).

Figure 2 shows the optimal cantilevers. As one would expect, the flexural rigidity of the beam increases when approaching the clamped end, but this is due to an increase in the Young modulus rather than in the cross-sectional area. In this case, the bending equations of the beam remain valid since the cantilever has a constant-area cross section.

The lightest homogeneous beam satisfying the maximum-deflection constraint for the specified load $P_0$ is the titanium-alloy cantilever of mass $F_0^* = 4.6$. For the optimal cantilever, the mass reduction is $(1 - F_0/F_0^*) \cdot 100\% = 35.6\%$ compared to the lightest homogeneous beam.

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